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A constructive comparison of the rings $R(X)$ and $R\langle X\rangle$ and application to the Lequain–Simis Induction Theorem

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Abstract

We constructively prove that for any ring $\mathbf{R}$ with Krull dimension $\leqslant d$, the ring $\mathbf{R}\langle X\rangle$ locally behaves like the ring $\mathbf{R}(X)$ or a localization of a polynomial ring of type $(S^{-1}\mathbf{R})[X]$ with S a multiplicative subset of $\mathbf{R}$ such that the Krull dimension of $S^{-1}\mathbf{R}$ is $\leqslant d-1$. As an application, we give a simple and constructive proof of the Lequain–Simis Induction Theorem which is an important variation of the Quillen Induction Theorem.

Keywords: Constructive mathematics; Lequain–Simis Induction Theorem; Quillen–Suslin Theorem; Finitely generated projective modules; Local–global principles; Prüfer domains; Arithmetical rings

Introduction

In this paper, we continue to follow the philosophy developed in the papers [1,4–6,9,11,17–24,26,27,30,36,37]. The main goal is to find the constructive content hidden in abstract proofs of concrete theorems in Commutative Algebra.

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The general method consists in replacing some abstract ideal objects whose existence is based on the third excluded middle principle and the axiom of choice by incomplete specifications of these objects. We think that this is a first step in the achievement of Hilbert's program for Abstract Algebra methods:

Hilbert's program. If we prove using ideal methods a concrete statement, one can always eliminate the use of these elements and obtain a purely elementary proof.

Constructive Algebra can be seen as an abstract version of Computer Algebra. In Computer Algebra, one tries to get efficient algorithms for solving "concrete problems given in an algebraic formulation." A problem is "concrete" if its hypotheses and conclusion do have a computational content.

Constructive Algebra can be understood as a first "preprocessing" for Computer Algebra: finding general algorithms, even if they are not efficient. Moreover, in Constructive Algebra one tries to give general algorithms for solving virtually "any" theorem of Abstract Algebra. So a first task is often to understand what is the computational content hidden in hypotheses that are formulated in a very abstract way. E.g., what is a good constructive definition for a local ring, a valuation ring, an arithmetical ring, a ring of Krull dimension $\leqslant 2$ and so on? A good constructive definition must be equivalent to the usual definition in classical mathematics, it has to have a computational content, and it has to be satisfied by usual objects (of usual mathematics) satisfying the abstract definition. See, e.g., Section 1.1.

Let us consider the classical theorem saying "any polynomial P in $\mathbf{K}[X]$ is a product of irreducible polynomials ($\mathbf{K}$ a field)." This leads to an interesting problem. Surely no general algorithm can give the solution of this theorem. So what is the constructive content of this theorem? A possible answer is the following one: when doing computations with P, you can always do as if you knew its decomposition in irreducibles. At the beginning, start as if P were irreducible. If some strange thing appears (the gcd of P and another polynomial Q is a strict divisor of P), use this fact in order to improve the decomposition of P.

This trick was invented in Computer Algebra as the D5-philosophy [8,31]. Following this computational trick you are able to compute inside the algebraic closure $\widetilde{\mathbf{K}}$ of $\mathbf{K}$ even if it is not possible to "construct" $\widetilde{\mathbf{K}}$.

This was called the "dynamical evaluation" (of the algebraic closure). And since our general method is directly inspired by this trick, we call it "constructive dynamical rereading of abstract proofs."

From a logical point of view, the "dynamical evaluation" gives a constructive substitute for two highly non-constructive tools of Abstract Algebra: the Third Excluded Middle, and Zorn's Lemma. These tools are needed to "construct" the algebraic closure $\widetilde{\mathbf{K}}$: the dynamical evaluation allows to find a fully computational content to this "construction."

In this paper, the dynamical evaluation is used in order to find constructive substitutes to very elegant abstract theorems as Quillen's patching, Quillen Induction and Lequain–Simis Induction.

Very important is the constructive rewriting of "abstract local–global principles." In classical proofs using this kind of principle, the argument is "let us see what happens after localization at an arbitrary prime ideal of $\mathbf{R}$." Prime ideals are too abstract objects from a computational point of view, particularly if you want to deal with a general commutative ring. In the constructive rereading, the argument is "let us see what happens when the ring is a residually discrete local ring," i.e., if $\forall x$, ($x \in \mathbf{R}^{\times}$ or $\forall y(1 + xy) \in \mathbf{R}^{\times}$). If you get a constructive proof in this particular case, you are done by "dynamically evaluating an arbitrary ring $\mathbf{R}$ as a residually discrete local ring." For more details see the "General Constructive Rereading Principle" in Section 1.1.

Let $\mathbf{R}$ be a commutative unitary ring. We denote by S (respectively, U) the multiplicative subset of $\mathbf{R}[X]$ formed by monic polynomials (respectively, primitive polynomials, i.e., polynomials whose coefficients generate the whole ring). Let

$$\mathbf{R}\langle X\rangle := S^{-1}\mathbf{R}[X] \quad \text{and} \quad \mathbf{R}(X) := U^{-1}\mathbf{R}[X].$$

The interest in the properties of $\mathbf{R}\langle X\rangle$ and $\mathbf{R}(X)$ branched in many directions and is attested by the abundance of articles on $\mathbf{R}\langle X\rangle$ and $\mathbf{R}(X)$ appearing in the literature (see [10] for a comprehensive list of papers dealing with the rings $\mathbf{R}\langle X\rangle$ and $\mathbf{R}(X)$). The ring $\mathbf{R}\langle X\rangle$ played an important role in Quillen's solution to Serre's conjecture [32] and its succeeding generalizations to non-Noetherian rings [2,16,28]. The construction $\mathbf{R}(X)$ turned out to be an efficient tool for proving results on $\mathbf{R}$ via passage to $\mathbf{R}(X)$.

It is clear that we have $\mathbf{R}[X] \subseteq \mathbf{R}\langle X\rangle \subseteq \mathbf{R}(X)$ and that $\mathbf{R}(X)$ is a localization of $\mathbf{R}\langle X\rangle$. The containment $\mathbf{R}\langle X\rangle \subseteq \mathbf{R}(X)$ becomes an equality if and only if $\mathbf{R}$ has Krull dimension 0 (in short, $\operatorname{Kdim}\mathbf{R} = 0$) [12].

In this paper, we will prove that for any ring $\mathbf{R}$ with Krull dimension $\leqslant d$, the ring $\mathbf{R}\langle X\rangle$ "dynamically behaves like the ring $\mathbf{R}(X)$ or a localization of a polynomial ring of type $(S^{-1}\mathbf{R})[X]$ with S a multiplicative subset of $\mathbf{R}$ and the Krull dimension of $S^{-1}\mathbf{R}$ is $\leqslant d-1$."

Recall that a module M over $\mathbf{R}[X_1,\dots,X_n] = \mathbf{R}[\underline{X}]$ is said to be *extended from* $\mathbf{R}$ (or simply, *extended*) if it is isomorphic to a module $N \otimes_{\mathbf{R}} \mathbf{R}[\underline{X}]$ for some $\mathbf{R}$-module N. Necessarily

$$N \simeq \mathbf{R} \otimes_{\mathbf{R}[\underline{X}]} M \quad \text{through} \quad \rho : \mathbf{R}[\underline{X}] \to \mathbf{R}, \quad f \mapsto f(0),$$

i.e., $N \simeq M/(X_1M + \cdots + X_nM)$. In particular, if M is finitely presented, denoting by $M^0 = M[0,\dots,0]$ the $\mathbf{R}$-module obtained by replacing the X_i by 0 in a relation matrix of M, then M is extended if and only if

$$M \simeq M^0 \otimes_{\mathbf{R}} \mathbf{R}[\underline{X}].$$

From a constructive point of view, a finitely generated projective module P is given by an idempotent matrix F such that $\operatorname{Im} F \simeq P$. If $F = F(\underline{X}) \in \mathbf{R}[\underline{X}]^{n\times n}$ (with $F^2 = F$) defines a projective module P over $\mathbf{R}[\underline{X}]$, then $P^0 \simeq \operatorname{Im}(F(0))$. Proving that the module P is extended from $\mathbf{R}$ (respectively, free) amounts to prove that the matrix F is conjugate to $F(0)$ (respectively, to a standard projection matrix).

In 1955, J.-P. Serre remarked [33] that it was not known whether there exist finitely generated projective modules over $\mathbf{A} = \mathbf{K}[X_1,\dots,X_k]$, $\mathbf{K}$ a field, which are not free. This remark turned into the "Serre conjecture," stating that indeed there were no such modules. Proven independently by D. Quillen [32] and A.A. Suslin [35], it became subsequently known as the Quillen–Suslin Theorem ([14] is an excellent exposition which has been updated recently in [15]). In [2,28], Maroscia and Brewer and Costa generalized the Quillen–Suslin Theorem to Prüfer domains with Krull dimension $\leqslant 1$. They proved that finitely generated projective modules over a polynomial ring with coefficients in a Prüfer domain $\mathbf{R}$ with Krull dimension $\leqslant 1$ are extended from $\mathbf{R}$. This result was a remarkable generalization of the Quillen–Suslin Theorem as it is free of any Noetherian hypothesis. The restriction to Prüfer domains with Krull dimension $\leqslant 1$ is due to the fact that $\mathbf{R}\langle X\rangle$ is a Prüfer domain if and only if $\mathbf{R}$ is a Prüfer domain with Krull dimension $\leqslant 1$.

Subsequently, in order to generalize the Quillen–Suslin Theorem to Prüfer domains and seeing that the class of Prüfer domains is not stable under the formation $\mathbf{R}\langle X\rangle$, Lequain and Simis [16] found a clever way to bypass this difficulty by proving the following new Induction Theorem.

Lequain–Simis Induction Theorem. *Suppose that a class of rings $\mathcal{F}$ satisfies the following properties*:

(i) *If $\mathbf{R} \in \mathcal{F}$, then every non-maximal prime ideal of $\mathbf{R}$ has finite height.*
(ii) $\mathbf{R} \in \mathcal{F} \Rightarrow \mathbf{R}[X]_{\mathfrak{p}[X]} \in \mathcal{F}$ *for any prime ideal $\mathfrak{p}$ of $\mathbf{R}$.*
(iii) $\mathbf{R} \in \mathcal{F} \Rightarrow \mathbf{R}_{\mathfrak{p}} \in \mathcal{F}$ *for any prime ideal $\mathfrak{p}$ of $\mathbf{R}$.*
(iv) $\mathbf{R} \in \mathcal{F}$ *and $\mathbf{R}$ local $\Rightarrow$ any finitely generated projective module over $\mathbf{R}[X]$ is free.*

Then, for each $\mathbf{R} \in \mathcal{F}$, if M is a finitely generated projective $\mathbf{R}[X_1, \dots, X_n]$-module, then M is extended from $\mathbf{R}$.

As an application of our dynamical comparison between the rings $\mathbf{R}(X)$ and $\mathbf{R}\langle X\rangle$, we give a constructive variation of Lequain–Simis Induction Theorem—using a simple proof. Note that Lequain and Simis put considerable effort for proving this marvellous theorem and they used some quite complicated technical steps.

Constructive Induction Theorem. *Let $\mathcal{F}$ be a class of commutative rings with finite Krull dimensions satisfying the properties below*:

(ii′) *If $\mathbf{R} \in \mathcal{F}$ then $\mathbf{R}(X) \in \mathcal{F}$.*
(iii) $\mathbf{R} \in \mathcal{F} \Rightarrow \mathbf{R}_S \in \mathcal{F}$ *for each multiplicative subset S in $\mathbf{R}$.*
(iv′) *If $\mathbf{R} \in \mathcal{F}$ then any finitely generated projective module over $\mathbf{R}[X]$ is extended from $\mathbf{R}$.*

Then, for each $\mathbf{R} \in \mathcal{F}$, if M is a finitely generated projective $\mathbf{R}[X_1, \dots, X_n]$-module, then M is extended from $\mathbf{R}$.

It is worth pointing out that when coupled with a result by Simis and Vasconcelos [34] asserting that over a valuation ring $\mathbf{V}$, all projective $\mathbf{V}[X]$-modules are free, the Lequain–Simis Induction Theorem yields to the fact that for any Prüfer domain $\mathbf{R}$, all finitely generated projective $\mathbf{R}[X_1, \dots, X_n]$-module are extended from $\mathbf{R}$.

For the purpose to prepare the ground for the generalizations of the Quillen–Suslin Theorem quoted above, we will give in Section 2 a constructive proof of a non-Noetherian version of the Quillen–Suslin Theorem for zero-dimensional rings. As a matter of fact, we will prove constructively that for any zero-dimensional ring $\mathbf{R}$, all finitely generated projective $\mathbf{R}[X_1, \dots, X_n]$-modules of constant rank are free (the Quillen–Suslin Theorem corresponding to the particular case $\mathbf{R}$ is a field). Note that there is no such constructive proof in the literature.

The undefined terminology is standard as in [13,15], and, for Constructive Algebra in [25,29].

1. A dynamical comparison between the rings $\mathbf{R}(X)$ and $\mathbf{R}\langle X\rangle$

1.1. Constructive preliminaries

If S is a multiplicative subset of a ring $\mathbf{R}$, the localization of $\mathbf{R}$ at S is the ring $S^{-1}\mathbf{R} = \{\frac{x}{s}, x \in \mathbf{R}, s \in S\}$ in which the elements of S are forced into being invertible. For $x_1, \dots, x_r \in \mathbf{R}$,

$\mathcal{M}(x_1, \dots, x_r)$ will denote the multiplicative subset of $\mathbf{R}$ generated by $x_1, \dots, x_r$, that is,

$$\mathcal{M}(x_1, \dots, x_r) = \{x_1^{n_1} \cdots x_r^{n_r}, n_i \in \mathbb{N}\}.$$

The localization of $\mathbf{R}$ at $\mathcal{M}(x_1, \dots, x_r)$ is the same one as the localization at $\mathcal{M}(x_1 \cdots x_r)$. If $x \in \mathbf{R}$, the localization of $\mathbf{R}$ at the multiplicative subset $\mathcal{M}(x)$ will be denoted by $\mathbf{R}_x$.

Definition 1.1 *(Comaximal multiplicative subsets [9]).* If $S_1, \dots, S_k$ are multiplicative subsets of $\mathbf{R}$, we say that $S_1, \dots, S_k$ are *comaximal* if

$$\forall s_1 \in S_1, \dots, s_n \in S_n, \exists a_1, \dots, a_n \in \mathbf{R} \text{ such that } \sum_{i=1}^{n} a_i s_i = 1.$$

Remark that comaximal multiplicative sets remain comaximal when you replace the ring by a bigger one or the multiplicative subsets by smaller ones.

Definition 1.2 *(Constructive definition of the radical).* Constructively, the *radical* $\mathrm{Rad}(\mathbf{R})$ of a ring $\mathbf{R}$ is the set of all the $x \in \mathbf{R}$ such that $1 + x\mathbf{R} \subset \mathbf{R}^\times$, where $\mathbf{R}^\times$ is the group of units of $\mathbf{R}$. A ring $\mathbf{R}$ is *local* if it satisfies:

$$\forall x \in \mathbf{R} x \in \mathbf{R}^\times \vee 1 + x \in \mathbf{R}^\times. \tag{1}$$

It is *residually discrete local* if it satisfies:

$$\forall x \in \mathbf{R} x \in \mathbf{R}^\times \vee x \in \mathrm{Rad}(\mathbf{R}). \tag{2}$$

From a classical point of view, we have (1) $\Leftrightarrow$ (2), but the constructive meaning of (2) is stronger than that of (1). Constructively a *discrete field* is defined as a ring in which each element is zero or invertible, with an explicit test for the "or." A *Heyting field* (or a field) is defined as a local ring whose Jacobson radical is 0. So $\mathbf{R}$ is residually discrete local exactly when it is local and the residue field $\mathbf{R}/\mathrm{Rad}(\mathbf{R})$ is a discrete field.

Definition 1.3 *(Constructive definition of Krull dimension [4,7,19]).* A ring $\mathbf{R}$ is said to have Krull dimension less or equal to d (in short, $\mathrm{Kdim}\,\mathbf{R} \leqslant d$) if for every $x \in \mathbf{R}$, $\mathrm{Kdim}\, S_{\mathbf{R},x}^{-1}\mathbf{R} \leqslant d - 1$, where $S_{\mathbf{R},x} = \{x^k(1 + yx), k \in \mathbb{N}, y \in \mathbf{R}\}$ and with the initialization $\mathrm{Kdim}\,\mathbf{R} \leqslant -1$ if $1 = 0$ in $\mathbf{R}$ ($\mathbf{R}$ is trivial). A ring $\mathbf{R}$ is said to be finite-dimensional if $\mathrm{Kdim}\,\mathbf{R} \leqslant d$ for some $d \in \mathbb{N}$.

As a particular case, if $\mathrm{Kdim}\,\mathbf{R} \leqslant d$, $d \geqslant 0$ and $x \in \mathrm{Rad}(\mathbf{R})$ then, constructively, $\mathrm{Kdim}\,\mathbf{R}[1/x] \leqslant d - 1$.

Let us note that we have only given a constructive definition for the sentence $\mathrm{Kdim}\,\mathbf{R} \leqslant d$. From a classical point of view this is sufficient since we can define $\mathrm{Kdim}\,\mathbf{R}$ as the least d possible. But this does not define a natural number in the constructive meaning. In fact, from a constructive point of view, it seems that the "restricted" definition is always sufficient for doing good mathematics.

An *integral domain* is a ring in which each element a is zero or regular (i.e., $ax = 0$ implies $x = 0$), with an explicit test for the "or."

Classically an arithmetical ring is a ring in which finitely generated ideals are locally principal. This is equivalent to the following constructive definition.

Definition 1.4 *(Constructive definition of arithmetical rings [9]).* A ring $\mathbf{R}$ is said to be *arithmetical* if for any $a, b \in \mathbf{R}$ there is a u such that $\langle a, b\rangle = \langle a\rangle$ in $\mathbf{R}_u$ and $\langle a, b\rangle = \langle b\rangle$ in $\mathbf{R}_{1-u}$. A *Prüfer ring* is a reduced arithmetical ring. A *Prüfer domain* is an arithmetical ring which is an integral domain. A *valuation domain* is a local Prüfer domain.

Valuation domains are also characterized as integral domains such that for any a, b, a divides b or b divides a (with an explicit test for the "or" and an explicit divisibility for "divides").

The General Constructive Rereading Principle

Let us now recall a General Constructive Rereading Principle which enables to automatically obtain a "quasi-global" version of a theorem from its local version.

Let I and U be two subsets of $\mathbf{R}$. We denote by $\mathcal{M}(U)$ the monoid generated by U, $\mathcal{I}_{\mathbf{R}}(I)$ or $\mathcal{I}(I)$ the ideal generated by I, and $\mathcal{S}_{\mathbf{R}}(I; U)$ or $\mathcal{S}(I; U)$ the monoid $\mathcal{M}(U) + \mathcal{I}_{\mathbf{R}}(I)$. If $I = \{a_1, \dots, a_k\}$ and $U = \{u_1, \dots, u_\ell\}$, we denote $\mathcal{M}(U)$, $\mathcal{I}(I)$ and $\mathcal{S}(I; U)$ respectively by $\mathcal{M}(u_1, \dots, u_\ell)$, $\mathcal{I}(a_1, \dots, a_k)$ and $\mathcal{S}(a_1, \dots, a_k; u_1, \dots, u_\ell)$.

Note that in the ring $S^{-1}\mathbf{R}$, where $S = \mathcal{S}(a_1, \dots, a_k; u_1, \dots, u_\ell)$, the u_j's are invertible and the a_i's are in the Jacobson radical. Moreover it is easy to see that for any $a \in \mathbf{R}$, the monoids $\mathcal{S}(I; U, a)$ and $\mathcal{S}(I, a; U)$ are comaximal in $\mathbf{R}_{\mathcal{S}(I;U)}$. These two remarks lead to the desired rereading principle.

General Principle 5 of [24]. When rereading an explicit proof given in case $\mathbf{R}$ is residually discrete local, with an arbitrary ring $\mathbf{R}$, start with $\mathbf{R} = \mathbf{R}_{\mathcal{S}(0;1)}$. Then, at each disjunction (for an element a produced when computing in the local case)

$$a \in \mathbf{R}^\times \vee a \in \operatorname{Rad}(\mathbf{R}),$$

replace the "current" ring $\mathbf{R}_{\mathcal{S}(I;U)}$ by both $\mathbf{R}_{\mathcal{S}(I;U,a)}$ and $\mathbf{R}_{\mathcal{S}(I,a;U)}$ in which the computations can be pursued. At the end of this rereading, one obtains a finite family of rings $\mathbf{R}_{\mathcal{S}(I_j;U_j)}$ with comaximal monoids $\mathcal{S}(I_j; U_j)$ and finite sets I_j, U_j.

1.2. The rings $\mathbf{R}(X)$ and $\mathbf{R}\langle X\rangle$

By the following theorem, we prove that for any ring $\mathbf{R}$ with Krull dimension $\leqslant d$, the ring $\mathbf{R}\langle X\rangle$ "dynamically behaves like the ring $\mathbf{R}(X)$ or a localization of a polynomial ring of type $(S^{-1}\mathbf{R})[X]$ with S a multiplicative subset of $\mathbf{R}$ and the Krull dimension of $S^{-1}\mathbf{R}$ is $\leqslant d-1$."

Theorem 1.5. *Let $d \in \mathbb{N}$ and $\mathbf{R}$ a ring with Krull dimension $\leqslant d$. Then for any primitive polynomial $f \in \mathbf{R}[X]$, there exist comaximal subsets $V_1, \dots, V_s$ of $\mathbf{R}\langle X\rangle$ such that for each $1 \leqslant i \leqslant s$, either f is invertible in $\mathbf{R}\langle X\rangle_{V_i}$ or $\mathbf{R}\langle X\rangle_{V_i}$ is a localization of $(S_{\mathbf{R},a_i}^{-1}\mathbf{R})[X]$, where $S_{\mathbf{R},a_i} = a_i^{\mathbb{N}}(1 + a_i\mathbf{R})$, for some coefficient a_i of f (note that* $\operatorname{Kdim} S_{\mathbf{R},a_i}^{-1}\mathbf{R} \leqslant d-1$).

Proof. *First case*: $\mathbf{R}$ *is residually discrete local*. Observe that any primitive polynomial $f \in \mathbf{R}[X]$ can be written in the form $f = g + u$ where $g, u \in \mathbf{R}[X]$, all the coefficients of g are in the Jacobson radical $\operatorname{Rad}(\mathbf{R})$ of $\mathbf{R}$ and u is quasi-monic (that is, the leading coefficient of u is invertible). If the degree of u is k, then $g = \sum_{j>k} a_j X^j$. Now we open two branches: we localize $\mathbf{R}\langle X\rangle$ at the comaximal multiplicative subsets generated by f and g.

$$\begin{tikzcd} & \mathbf{R}\langle X\rangle \arrow[dl, dash] \arrow[dr, dash] & \\ \mathbf{R}\langle X\rangle_f & & \mathbf{R}\langle X\rangle_g \end{tikzcd}$$

In $\mathbf{R}\langle X\rangle_f$, f is clearly invertible.

In $\mathbf{R}\langle X\rangle_g$, write $g = \sum_{j=k+1}^{m} a_j X^j$, where the $a_j \in \mathrm{Rad}(\mathbf{R})$. It follows that the multiplicative subsets $\mathcal{M}(a_{k+1}), \dots, \mathcal{M}(a_s)$ are comaximal in $\mathbf{R}\langle X\rangle_g$. Note that for any $k+1 \leqslant i \leqslant m$, $\mathcal{M}(a_i)^{-1}(\mathbf{R}\langle X\rangle_g)$ is a localization of the polynomial ring $\mathbf{R}_{a_i}[X]$ and $\dim \mathbf{R}_{a_i} < \dim \mathbf{R}$.

$$\begin{tikzcd} & \mathbf{R}\langle X\rangle_g \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ \mathbf{R}\langle X\rangle_{a_{k+1}} & \cdots & \mathbf{R}\langle X\rangle_{a_m} \end{tikzcd}$$

General case: $\mathbf{R}$ *arbitrary*. Apply the General Constructive Rereading Principle. Precisely this gives the following computation. First we remark that since f is primitive, say $f = \sum_{j=0}^{m} a_j X^j$, the multiplicative subsets $U_m = \mathcal{M}(a_m)$, $U_{m-1} = \mathcal{S}_{\mathbf{R}}(a_m; a_{m-1})$, $\dots$, $U_k = \mathcal{S}_{\mathbf{R}}(a_m, \dots, a_{k+1}; a_k)$, $\dots$, $U_0 = \mathcal{S}_{\mathbf{R}}(a_m, \dots, a_1; a_0)$ are comaximal in $\mathbf{R}$. It is now sufficient to prove the conclusion for each ring $\mathbf{R}_{U_i}$. And this conclusion is obtained from the proof given for the first case. $\square$

Remark 1.6. If $\mathbf{R}$ is a valuation domain then any $f \in \mathbf{R}[X]$ is easily written as $f = ag$ where $a \in \mathbf{R}$ and $g \in \mathbf{R}[X]$ is primitive, invertible in $\mathbf{R}(X)$. From this fact, it follows easily that $\mathbf{R}(X)$ is again a valuation domain, and if $\mathrm{Kdim}\,\mathbf{R} \leqslant d$ then $\mathrm{Kdim}\,\mathbf{R}(X) \leqslant d$. So by Theorem 1.5, we painlessly get constructively that:

(i) If $\mathbf{R}$ is a valuation domain with $\mathrm{Kdim}\,\mathbf{R} \leqslant 1$ then $\mathbf{R}\langle X\rangle$ is a Prüfer domain with $\mathrm{Kdim}\,\mathbf{R} \leqslant 1$. As a matter of fact, it is clear that in this case, in one of the $\mathbf{R}\langle X\rangle_{U_i}$, the computations are done like in $\mathbf{R}(X)$, while the other $\mathbf{R}\langle X\rangle_{U_i}$ are localizations of the polynomial ring $\mathbf{K}[X]$ where $\mathbf{K}$ is the quotient field of $\mathbf{R}$.

(ii) If $\mathbf{R}$ is a Prüfer domain with $\mathrm{Kdim}\,\mathbf{R} \leqslant 1$ then so is $\mathbf{R}\langle X\rangle$ (the Maroscia–Brewer–Costa Theorem [2,28]). This is obtained from (i) by application of the General Constructive Rereading Principle.

Remark 1.7. If $\mathrm{Kdim}\,\mathbf{R} = 0$ then clearly $\mathbf{R}\langle X\rangle = \mathbf{R}(X)$ (the rings $S_{a_i}^{-1}\mathbf{R}$ in Theorem 1.5 being trivial).

2. A constructive proof of the Quillen–Suslin Theorem

We recall here the main steps of the constructive proof obtained in [1,26] by deciphering Quillen's proof of the Quillen–Suslin Theorem (a slightly more involved constructive deciphering was first given in [24]).

2.1. The patchings of Quillen and Vaserstein

We will state the following theorem without proof. Constructive proofs can be found in [1,26].

Theorem 2.1 *(Vaserstein's patching, constructive form). Let M be a matrix in $\mathbf{R}[X]$ and consider $S_1, \dots, S_n$ comaximal multiplicative subsets of $\mathbf{R}$. Then $M(X)$ and $M(0)$ are equivalent over $\mathbf{R}[X]$ if and only if, for each $1 \leqslant i \leqslant n$, they are equivalent over $\mathbf{R}_{S_i}[X]$.*

Theorem 2.2 *(Quillen's patching, constructive form). Let P be a finitely presented module over $\mathbf{R}[X]$ and consider $S_1, \dots, S_n$ comaximal multiplicative subsets of $\mathbf{R}$. Then P is extended from $\mathbf{R}$ if and only if for each $1 \leqslant i \leqslant n$, P_{S_i} is extended from $\mathbf{R}_{S_i}$.*

Proof. This is a corollary of the previous theorem since the isomorphism between $P(X)$ and $P(0)$ is nothing but the equivalence of two matrices $A(X)$ and $A(0)$ constructed from a relation matrix $M \in \mathbf{R}^{q\times m}$ of $P \simeq \operatorname{Coker} M$ (see [13]):

$$A(X) = \begin{bmatrix} M(X) & 0_{q,q} & 0_{q,q} & 0_{q,m} \\ 0_{q,m} & \mathrm{I}_q & 0_{q,q} & 0_{q,m} \end{bmatrix}. \qquad \square$$

2.2. Horrocks' Theorem

Local Horrocks' Theorem is the following result.

Theorem 2.3 *(Local Horrocks Extension Theorem). If $\mathbf{R}$ is a residually discrete local ring and P a finitely generated projective module over $\mathbf{R}[X]$ which is free over $\mathbf{R}\langle X\rangle$, then it is free over $\mathbf{R}[X]$ (i.e., extended from $\mathbf{R}$).*

Note that the hypothesis $M \otimes_{\mathbf{R}[X]} \mathbf{R}\langle X\rangle$ is a free $\mathbf{R}\langle X\rangle$-module is equivalent to the fact that M_f is a free $\mathbf{R}[X]_f$-module for some monic polynomial $f \in \mathbf{R}[X]$ (see, e.g., Corollary 2.7, p. 18 in [15]). The detailed proof given by Kunz [13] is elementary and constructive, except Lemma 3.13 whose proof is abstract since it uses maximal ideals. In fact this lemma asserts if P is a projective module over $\mathbf{R}[X]$ which becomes free of rank k over $\mathbf{R}\langle X\rangle$, then its kth Fitting ideal equals $\langle 1\rangle$. This result has the following elementary constructive proof. If $P \oplus Q \simeq \mathbf{R}[X]^m$ then $P \oplus Q_1 = P \oplus (Q \oplus \mathbf{R}[X]^k)$ becomes isomorphic to $\mathbf{R}\langle X\rangle^{m+k}$ over $\mathbf{R}\langle X\rangle$ with Q_1 isomorphic to $\mathbf{R}\langle X\rangle^m$ over $\mathbf{R}\langle X\rangle$. So we may assume $P \simeq \operatorname{Im} F$, where $G = \mathrm{I}_n - F \in \mathbf{R}[X]^{n\times n}$ is an idempotent matrix, conjugate to a standard projection matrix of rank $n-k$ over $\mathbf{R}\langle X\rangle$. We deduce that $\det(\mathrm{I}_n + TG) = (1+T)^{n-k}$ over $\mathbf{R}\langle X\rangle$. Since $\mathbf{R}[X]$ is a subring of $\mathbf{R}\langle X\rangle$ this remains true over $\mathbf{R}[X]$. So the sum of all $n-k$ principal minors of G is equal to 1 (i.e. the coefficient of T^{n-k} in $\det(\mathrm{I}_n + TG)$). Hence we conclude by noticing that G is a relation matrix for P. For more details see, e.g., [25].

A global version is obtained from a constructive proof of the local one by the Quillen's patching and applying the General Constructive Rereading Principle.

Theorem 2.4 *(Global Horrocks extension theorem). Let S be the multiplicative set of monic polynomials in $\mathbf{R}[X]$, $\mathbf{R}$ an arbitrary commutative ring. If P is a finitely generated projective module over $\mathbf{R}[X]$ such that P_S is extended from $\mathbf{R}$, then P is extended from $\mathbf{R}$.*

2.3. Quillen Induction

Classical Quillen Induction is the following one.

Quillen Induction. *Suppose that a class of rings $\mathcal{P}$ satisfies the following properties:*

(i) *If $\mathbf{R} \in \mathcal{P}$ then $\mathbf{R}\langle X\rangle \in \mathcal{P}$.*
(ii) *If $\mathbf{R} \in \mathcal{P}$ then $\mathbf{R}_{\mathfrak{m}} \in \mathcal{P}$ for any maximal ideal $\mathfrak{m}$ of $\mathbf{R}$.*
(iii) *If $\mathbf{R} \in \mathcal{P}$ and $\mathbf{R}$ is local, and if M is a finitely generated projective $\mathbf{R}[X]$-module, then M is extended from $\mathbf{R}$ (that is, free).*

Then, for each $\mathbf{R} \in \mathcal{P}$, if M is a finitely generated projective $\mathbf{R}[X_1, \dots, X_n]$-module, then M is extended from $\mathbf{R}$.

Quillen Induction needs maximal ideals, it works in classical mathematics but it cannot be fully constructive. The fact that (ii) and (iii) imply the case $n = 1$ in the conclusion needs a priori a constructive rereading, where one replaces Quillen's patching with maximal ideals by the constructive form (Theorem 2.2) with comaximal multiplicative subsets.

On the contrary, the "inductive step" in the proof is elementary (see, e.g., [15]) and is based only on the following hypotheses.

(i) If $\mathbf{R} \in \mathcal{P}$ then $\mathbf{R}\langle X\rangle \in \mathcal{P}$.
(iii′) If $\mathbf{R} \in \mathcal{P}$ and M is a finitely generated projective $\mathbf{R}[X]$-module, then M is extended from $\mathbf{R}$.

In the case of Serre's problem, $\mathbf{R}$ is a discrete field. So (i) and (iii′) are well known. Remark that (iii′) is also given by Horrocks' global theorem. So Quillen's proof is deciphered in a fully constructive way. Moreover, since a zero-dimensional reduced local ring is a discrete field we obtain the following well-known generalization (see [2]).

Theorem 2.5 *(Quillen–Suslin, non-Noetherian version).*

(1) *If $\mathbf{R}$ is a zero-dimensional reduced ring then any finitely generated projective module P over $\mathbf{R}[X_1, \dots, X_n]$ is extended from $\mathbf{R}$ (i.e., isomorphic to a direct sum of modules $e_i\mathbf{R}[\underline{X}]$ where the e_i's are idempotent elements of $\mathbf{R}$).*
(2) *As a particular case, any finitely generated projective module of constant rank over $\mathbf{R}[X_1, \dots, X_n]$ is free.*
(3) *More generally the results work for any zero-dimensional ring.*

Proof. The first point can be obtained from the local case by the constructive Quillen's patching. It can also be viewed as a concrete application of the General Constructive Rereading Principle.

Let us denote by $\mathbf{R}_{\mathrm{red}}$ the reduced ring associated to a ring $\mathbf{R}$. Recall that $\mathrm{GK}_0(\mathbf{R})$ is the set isomorphism classes of finitely generated projective $\mathbf{R}$-modules.

The third point follows from the fact that the canonical map $M \mapsto M_{\mathrm{red}}, \mathrm{GK}_0(\mathbf{R}) \to \mathrm{GK}_0(\mathbf{R}_{\mathrm{red}})$ is a bijection. Moreover $\mathbf{R}_{\mathrm{red}}[X_1, \ldots, X_n] = \mathbf{R}[X_1, \ldots, X_n]_{\mathrm{red}}$. $\square$

3. The Lequain–Simis Induction Theorem

In order to generalize the Quillen–Suslin Theorem to Prüfer domains and seeing that the class of Prüfer domains is not stable under the formation $\mathbf{R}\langle X\rangle$, Lequain and Simis [16] found a clever way to bypass this difficulty by proving the following new Induction Theorem.

Lequain–Simis Induction. *Suppose that a class of rings $\mathcal{F}$ satisfies the following properties*:

(i) *If $\mathbf{R} \in \mathcal{F}$, then every non-maximal prime ideal of $\mathbf{R}$ has finite height.*
(ii) $\mathbf{R} \in \mathcal{F} \Rightarrow \mathbf{R}[X]_{\mathfrak{p}[X]} \in \mathcal{F}$ *for any prime ideal $\mathfrak{p}$ of $\mathbf{R}$.*
(iii) $\mathbf{R} \in \mathcal{F} \Rightarrow \mathbf{R}_{\mathfrak{p}} \in \mathcal{F}$ *for any prime ideal $\mathfrak{p}$ of $\mathbf{R}$.*
(iv) $\mathbf{R} \in \mathcal{F}$ *and $\mathbf{R}$ local $\Rightarrow$ any finitely generated projective module over $\mathbf{R}[X]$ is free.*

Then, for each $\mathbf{R} \in \mathcal{F}$, if M is a finitely generated projective $\mathbf{R}[X_1, \ldots, X_n]$-module, then M is extended from $\mathbf{R}$.

Note here that if $\mathbf{R}$ is local with maximal ideal $\mathfrak{m}$, then $\mathbf{R}(X) = \mathbf{R}[X]_{\mathfrak{m}[X]}$.

We propose here a constructive variation of Lequain–Simis Induction Theorem using a simple proof. This is one important application of our dynamical comparison between the rings $\mathbf{R}(X)$ and $\mathbf{R}\langle X\rangle$.

Theorem 3.1 *(Constructive Induction Theorem). Let $\mathcal{F}$ be a class of commutative rings with finite Krull dimensions satisfying the properties below*:

(ii′) *If $\mathbf{R} \in \mathcal{F}$ then $\mathbf{R}(X) \in \mathcal{F}$.*
(iii) $\mathbf{R} \in \mathcal{F} \Rightarrow \mathbf{R}_S \in \mathcal{F}$ *for each multiplicative subset S in $\mathbf{R}$.*
(iv′) *If $\mathbf{R} \in \mathcal{F}$ then any finitely generated projective module over $\mathbf{R}[X]$ is extended from $\mathbf{R}$.*

Then, for each $\mathbf{R} \in \mathcal{F}$, if M is a finitely generated projective $\mathbf{R}[X_1, \ldots, X_n]$-module, then M is extended from $\mathbf{R}$.

Proof. We reason by double induction on the number n of variables and the Krull dimension of the basic ring $\mathbf{R}$.

For the initialization of the induction there is no problem since if $n = 1$ there is nothing to prove and for polynomial rings over zero-dimensional rings (see Theorem 2.5) the result is true constructively.

We assume that the construction is given with n variables for rings in $\mathcal{F}$. Then we consider the case of $n + 1$ variables and we give the proof by induction on the dimension of the ring $\mathbf{R} \in \mathcal{F}$. We assume that the dimension is $\leqslant d + 1$ with $d \geqslant 0$ and the construction has been done for rings of dimension $\leqslant d$.

Let P be a finitely generated projective $\mathbf{R}[X_1, \ldots, X_n, Y]$-module. Let us denote X for $X_1, \ldots, X_n$. The module P can be seen as the cokernel of a presentation matrix $M = M(X, Y)$ with entries in $\mathbf{R}[X, Y]$. Let $A(X, Y)$ be the associated enlarged matrix (as in the proof of Theorem 2.2).

Using the induction hypothesis over n and (ii′) we know that $A(X, Y)$ and $A(0, Y)$ are equivalent over the ring $\mathbf{R}(Y)[X]$. This means that there exist matrices Q_1, R_1 with entries in $\mathbf{R}[X, Y]$ such that

$$Q_1 A(X, Y) = A(0, Y) R_1 \tag{3}$$

$$\text{where } \det(Q_1) \text{ and } \det(R_1) \text{ are primitive polynomials in } \mathbf{R}[Y]. \tag{4}$$

We first want to show that $A(X, Y)$ and $A(0, Y)$ are equivalent over $\mathbf{R}\langle Y\rangle[X]$. Using the Vaserstein's patching, for doing this job it is sufficient to show that A and $A(0, Y)$ are equivalent over $\mathbf{R}\langle Y\rangle[X]_{\mathcal{M}_i}$ for comaximal multiplicative subsets $\mathcal{M}_i$.

We consider the primitive polynomial $f = \det(Q_1)\det(R_1) \in \mathbf{R}[Y]$ and we apply Theorem 1.5. We get comaximal subsets $V_1, \ldots, V_s$ of $\mathbf{R}\langle Y\rangle$ such that for each $1 \leqslant i \leqslant s$, either f is invertible in $\mathbf{R}\langle Y\rangle_{V_i}$ or $\mathbf{R}\langle Y\rangle_{V_i}$ is a localization of $\mathbf{R}_{a_i}[Y]$ for some $a_i \in \mathbf{R}$ such that $\mathbf{R}_{a_i}$ has Krull dimension $\leqslant d$.

In the first case $\det(Q_1)$ and $\det(R_1)$ are invertible in $\mathbf{R}\langle Y\rangle_{V_i}$. This implies that $A(X, Y)$ and $A(0, Y)$ are equivalent over $\mathbf{R}\langle Y\rangle[X]_{Vi}$.

In the second case, by induction hypothesis over the dimension, $A(X, Y)$ and $A(0, 0)$ are equivalent over $\mathbf{R}_{a_i}[Y][X]$. An immediate consequence is that $A(X, Y)$ and $A(0, Y)$ are equivalent over $\mathbf{R}_{a_i}[Y][X]$. Finally they are also equivalent over $\mathbf{R}\langle Y\rangle[X]_{V_i}$ which is a localization of the previous ring.

Now we know that there exist invertible matrices Q, R over the ring $\mathbf{R}\langle Y\rangle[X] \subseteq (\mathbf{R}[X])\langle Y\rangle$ such that

$$QA(X, Y) = A(0, Y)R.$$

We know also that $A(0, 0)$ and $A(0, Y)$ are equivalent over $\mathbf{R}[Y] \subseteq (\mathbf{R}[X])\langle Y\rangle$ (case $n = 1$) and $A(0, 0)$ and $A(X, 0)$ are equivalent over $\mathbf{R}[X] \subseteq (\mathbf{R}[X])\langle Y\rangle$. So $A(X, 0)$ and $A(X, Y)$ are equivalent over $(\mathbf{R}[X])\langle Y\rangle$, and by virtue of global Horrocks' Theorem (Theorem 2.4), P is extended from $\mathbf{R}[X]$, i.e., $A(X, 0)$ and $A(X, Y)$ are equivalent over $\mathbf{R}[X, Y]$. By induction hypothesis, P is extended from $\mathbf{R}$. $\square$

Remark 3.2. In fact, the proof does not use "any" multiplicative subset of rings $\mathbf{R}$ in $\mathcal{F}$, but only multiplicative subsets obtained by iterating localizations at some $\mathcal{S}(a_1, \ldots, a_k; u)$.

Recall that a ring is called a pp-ring if the annihilator ideal of any element is generated by an idempotent.

Corollary 3.3 *(Lequain–Simis Theorem). For any finite-dimensional arithmetical pp-ring* $\mathbf{R}$*, all finitely generated projective* $\mathbf{R}[X_1, \ldots, X_n]$*-modules,* $n \geqslant 2$*, are extended from* $\mathbf{R}$ *if and only if all finitely generated projective* $\mathbf{R}[X_1]$*-modules are extended from* $\mathbf{R}$.

Proof. We prove that the class $\mathcal{F}$ of finite-dimensional arithmetical pp-rings such that all finitely generated projective $\mathbf{R}[X_1]$-modules are extended from $\mathbf{R}$ satisfies the hypothesis in our Induction Theorem. Only the first point (ii′) is problematic. We assume to have a constructive proof in the local case, i.e., the case of valuation domains. So, starting with an arithmetical pp-ring, the General Constructive Rereading Principle gives comaximal multiplicative sets where the needed computations are done successfully. This allows to give the desired global conclusion in an explicit way. □

Remark 3.4. Thierry Coquand announced recently a constructive proof of the Bass–Simis–Vasconcelos Theorem (projective modules over $\mathbf{V}[X]$, $\mathbf{V}$ a valuation domain, are free) [3].

As always constructive proofs work in classical mathematics and Theorem 3.1 applies. Moreover, in classical mathematics, we get the following variation:

Theorem 3.5 *(New Classical Induction Theorem). Let $\mathcal{F}$ be a class of commutative rings with finite Krull dimensions satisfying the properties below*:

- (ii) *If $\mathbf{R} \in \mathcal{F}$ and $\mathbf{R}$ is local then $\mathbf{R}(X) \in \mathcal{F}$.*
- (iii′) *$\mathbf{R} \in \mathcal{F} \Rightarrow \mathbf{R}_S \in \mathcal{F}$ for each multiplicative set S in $\mathbf{R}$.*
- (iv) *If $\mathbf{R} \in \mathcal{F}$ and $\mathbf{R}$ is local then any finitely generated projective module over $\mathbf{R}[X]$ is extended from $\mathbf{R}$.*

Then, for each $\mathbf{R} \in \mathcal{F}$, if M is a finitely generated projective $\mathbf{R}[X_1, \dots, X_n]$-module, then M is extended from $\mathbf{R}$.

Proof. From (ii) and (iv) we deduce (ii′) and (iv′) in Theorem 3.1 by using the abstract Quillen's patching that uses maximal ideals. □

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